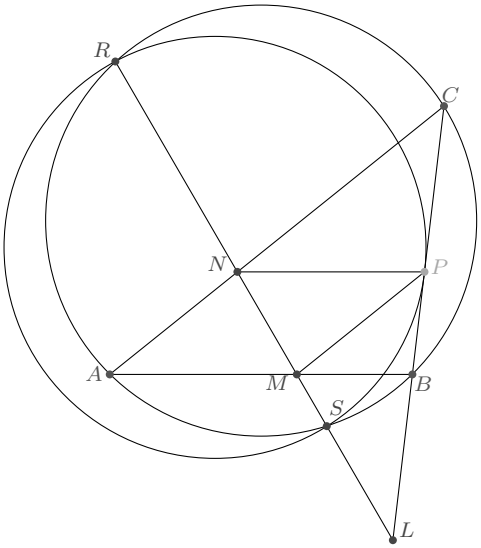
Concours Général de Mathétmatiques "Minko Balkanski"

Solutions

9 May 2015

Solution 1.

Let MN intersect BC at L. Applying Thales theorem to ΔLNB et ΔLNP we get $\frac{LP}{LB} = \frac{LM}{LN} = \frac{LC}{LP}$. Hence, $LP^2 = LC.LB = LR.LS$ (this is the power of L with respect to the circumcircle of ΔABC). Thus LP is tangent to the circumcircle of ΔRPS.



⋆

Solution 2.

30. Let n be an integer that verifies the condition. We order the primes $p_1, p_2, \ldots$. In other words, $p_1 = 2, p_2 = 3, p_3 = 5$ and so on. Let k be the smallest integer such that $p_k \nmid n$. We then have $p_k^2 > n$ as it is clearly co-prime with n. On the other hand, $p_1, \ldots, p_{k-1}$ divide n. It clearly follows that so does their product. We then have $p_1 p_2 \ldots p_{k-1} \leqslant n < p_k^2$. This inequality is clearly only possible for a finite number of k.

Any available theorem that lets us control the size of p_k can be used to finish the solution. For example, using Bertrand's postulate we have $p_k < 2p_{k-1}$ and then $p_k^2 < 4p_{k-1}^2 < 8p_{k-1}p_{k-2}$. This means that $p_1 p_2 \ldots p_{k-3} < 8$. We have $p_1 p_2 p_3 = 30 > 8$, therefore $k \leqslant 6$. By calculating directly we obtain $k \leqslant 4$. This proves that $n \leqslant 48$. What is more, if $n \geqslant 25 = p_3^2$, then 30 is a divisor of n, which leaves $n = 30$ as the only option. It is easy to check that $n = 30$ verifies the conditions (thus also eliminating the need to consider $n < 25$ as we are only interested in the largest n).

⋆

Solution 3.

Suppose that there are two non-constant polynomials Q and R with integer coefficients such that WLOG $\deg P \leqslant \frac{n}{2} \leqslant \deg Q$ and

$$P = (X - a_1) \dots (X - a_n) - 1 = Q.R\,.$$

Then $\forall i \in \{1, \dots n\}$, $Q(a_i)P(a_i) = -1$ and the values are ± 1. Then by the pigeon-hole principle P takes the same value at least $\frac{n}{2}$ times. If $\deg P < \frac{n}{2}$ we get that P is constant – contradiction. Then $\deg P = \deg Q = \frac{n}{2}$ and WLOG $P = (X - a_1) \dots (X - a_{n/2}) - 1$ and $Q = (X - a_{n/2+1}) \dots (X - a_n) - 1$, as those are the only non-constant polynomials of the right degree taking the value -1 on half of the a_i and 1 on the other half. However, $P = QR = P - Q - R$, so $Q = -P$, which is false looking at the leading coefficient.

$\star$

Solution 4.

1. We say that a point x is fixed if $\sigma(x) = x$. Take $x \in X$ that is not a fixed point. Consider $\sigma^n(x)$ for $n \in \mathbb{N}$. Its values are in X which is finite, therefore there are n and m such that $\sigma^n(x) = \sigma^m(x)$. We then have $\sigma^{n-m}(x) = x$ for some $n - m \in \mathbb{N}$. Let k be the minimal integer which satisfies this property and consider p defined by: $p(\sigma^i(x)) = \sigma^{i+1}(x)$ for $i = 0, 1, \dots, k-1$ and $p(y) = y$ for any other y. Note that since $\sigma^k(x) = x$, this defines a cycle. Then $\tilde{\sigma} = p^{-1} \circ \sigma$ is also a bijection, and for every i, $\tilde{\sigma}(\sigma^i(x)) = \sigma^{-1}(\sigma(\sigma^i(x))) = \sigma^i(x)$ is a fixed point. As all the points in the support of p can be written as $\sigma^i(x)$ for some i, we have that the support of $\tilde{\sigma}$ is strictly included in the support of σ and is disjoint with the support of p. The proof is completed by induction using a decomposition into cycles of $\tilde{\sigma}$ and multiplying it by p (and restricting $\tilde{\sigma}$ to the complement of the support of p to avoid overlapping).

2. The function θ is injective so $|\theta(D)| \geqslant |D|$. If its image is contained in D, it must be exactly D. Thus we can see θ as a permutation of D. By the previous question, it can be written as the product of disjoint cycles. If we prove the result for each cycle, by combining the sets of p_i we will obtain the result we seek. Therefore, we can assume that θ is the cycle $(a_1 a_2 \dots a_n)$ as defined in the exercise without loss of generality. As θ is not surjective, there is $b \in X \backslash D$. For $i = 1, \dots, n-1$ consider the application $p_i(a_i) = a_{i+1}$, $p_i(x) = x$ if $x \neq a_i$. It is easy to check that $p_i \circ p_i = p_i$. Let $p_n(a_n) = b$ and fixed on all other points and $p_{n+1}(b) = a_1$ and fixed on all other points. As $b \notin D$ we then have that $p_{n+1} \circ p_1 \circ p_2 \circ \dots \circ p_n$ coincides with θ on D.

3. Take an x that is not a fixed point. If there is $n > 0$ such that $\theta^n(x) = x$, then, as seen in the first question, we can write $\theta = p \circ \tilde{\theta}$. Here p is a cycle on the orbit of x and the support of $\tilde{\theta}$ is strictly included in the support of θ. We have already treated cycles in the last question, therefore we just need to consider $\tilde{\theta}$. In other words, we can assume without loss of generality that for every x that isn't a fixed point, $\theta^n(x) \neq x$ for every $n > 0$.

Let $A = \{x \in D | \nexists y \in D : \theta(y) = x\}$. For any point b in D that is not fixed, we can form the sequence $\theta^{-n}(b)$. On each step, $\theta^{-n}(b)$ is either in A, or is the image of some $y \in D$. From injectivity, if it is not in A it is the image of exactly one y, and we define that y to be $\theta^{-n-1}(b)$. As b is not fixed, those values are all different, and thus the sequence is finite. This shows that we can write $b = \theta^n(a)$ for some $a \in A$. It is easy to see that this representation is unique. Then $\theta = \prod_{a \in A} \theta_a$ where $\theta_a(\theta^k(a)) = \theta^{k+1}(a)$ for every k for which $\theta^k(a) \in D$, and $\theta_a(x) = x$ for any other $x \in D$. We can assume without loss of generality $A = \{a\}$ and thus $\theta = \theta_a$.

Let n be the largest integer such that $\theta^n(a) \in D$. For $k = 0, 1, \ldots, n$ we define p_k by $p_k(\theta^k(a)) = \theta^{k+1}(a)$ and fixed on all other points. We then have $\theta = p_0 \circ p_1 \circ \cdots \circ p_n$.

4. Let $Y = \{y \in X | \exists x \in X : \alpha(x) = y\}$. For any $y \in Y$ let $f(y)$ be arbitrary such that $\alpha(f(y)) = y$. This defines an injective function. For $x \in X$ such that $x \neq f(\alpha(x))$, define $p_x(x) = \alpha(x)$ and leave all the other points fixed for p_x. Then

$$\alpha = \prod_{x \neq f(\alpha(x))} p_x \circ \theta \,,$$

where θ coincides with α on $f(Y)$ and has all its other points fixed. In other words, θ's support is contained within points of the type $f(y)$ for some $y \in Y$, and is equal to α on these points, meaning its image is that same y. It is clear that $\theta\restriction_{f(Y)}$ is injective. What is more, the image and domain of α are the same, and finite, so as α is not bijective, it is not surjective. Therefore, Y is strictly included in X and thus so is $f(Y)$. In that case, we can apply the result from the previous question for $\theta\restriction_{f(Y)}$, which concludes the proof.

$\star$

Solution 5.

1. As $\alpha > 1$, we have, for every $k \in \mathbb{N}$, $\lfloor (n+1)\alpha \rfloor > (n+1)\alpha - 1 > n\alpha > \lfloor n\alpha \rfloor$. Those are then two different integers, and thus $\mathrm{Card}(\{\lfloor k\alpha \rfloor, k \in \mathbb{N}\} \cap \{1, \ldots, n\}) = \lfloor \frac{n}{\alpha} \rfloor$. Divided by n, this is then between $\frac{1}{\alpha} - \frac{1}{n}$ and $\frac{1}{\alpha}$, so it clearly converges towards α^{-1}.

2.

$a \Rightarrow b$ If α is a rational, $\alpha = \frac{p}{q}$, so is β, let it be $\frac{p'}{q'}$. But then $\lfloor p'q\alpha \rfloor = \lfloor pq'\beta \rfloor = pp'$ is in both sets which is a contradiction. Then both α and β are irrational and we can apply the first question. Moreover, it is easy to check that the sum of the densities of a partition of a set, if they are all defined, is equal to the density of the set. Hence, $\frac{1}{\alpha} + \frac{1}{\beta}$ is equal to the density of $\mathbb{N}$, which is 1.

$b \Rightarrow a$ First we check that E and F are disjoint. Assume the contrary, that is $k \in E \cap F$. Let $k = \lfloor n\alpha \rfloor = \lfloor m\beta \rfloor$. By definition we have $k \leqslant n\alpha < k+1$ and $k \leqslant m\beta < k+1$. We divide those by α and β respectively, obtaining $\frac{k}{\alpha} \leqslant n < \frac{k}{\alpha} + \frac{1}{\alpha}$ and $\frac{k}{\beta} \leqslant m < \frac{k}{\beta} + \frac{1}{\beta}$. By (b) this gives us $k \leqslant n + m < k + 1$. As all three are integers, this gives us an equality. Therefore we have equalities in the previous inequalities as well, but that is impossible as α and β are both irrational.

We now show that $E \cup F = \mathbb{N}$. Set $n \in \mathbb{N}$, $p = \lfloor n\alpha \rfloor$ and m the unique integer (possibly zero) such that $\lfloor m\beta \rfloor < p < \lfloor (m+1)\beta \rfloor$. In other words, we take $m = \lfloor \frac{p}{\beta} \rfloor$ and we have strict inequalities by what we just proved. We have that $\lfloor m\beta \rfloor$ is the largest element of F in $\{1, \ldots, p\}$. Similarly, $\lfloor n\alpha \rfloor = p$ is clearly the largest element of E in this set. Therefore E has exactly n elements in it, and F has exactly m. Therefore $E \cup F$ has exactly $n + m$ elements in $\{1, \ldots, p\}$. From the inequalities we have on p one deduces $\frac{p}{\alpha} \leqslant n < \frac{p+1}{\alpha}$ and $\frac{p}{\beta} - 1 < m < \frac{p+1}{\beta}$. Therefore, $p - 1 < m + n < p + 1$ and as those are all integers, $n + m = p$. In other words, every element of $\{1, \ldots, \lfloor n\alpha \rfloor\}$ is in $E \cup F$. This proves the result setting n go grow to infinity.

$\star$