Concours Général de Mathétmatiques "Minko Balkanski"

Solutions

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Solution 1.

The problem is equivalent to proving that

$$p^3 \mid (2p)\dots(p+1) - 2.p!.$$

For $p = 2$ this is trivial. Assume $p \neq 2$. Hence, it suffices to prove that

$$p^2 \mid (p+(p-1))\dots(p+1) - (p-1)!.$$

Expanding this expression partially modulo p^2 we get

$$(p-1)! + p((p-1)+\dots 1) - (p-1)! = \frac{p-1}{2}p^2,$$

which concludes the proof.

Remark One can also use the identity

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{n-k}.$$

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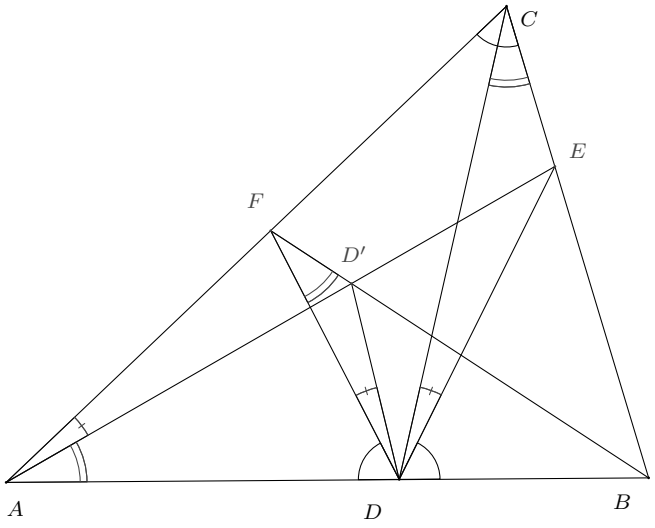
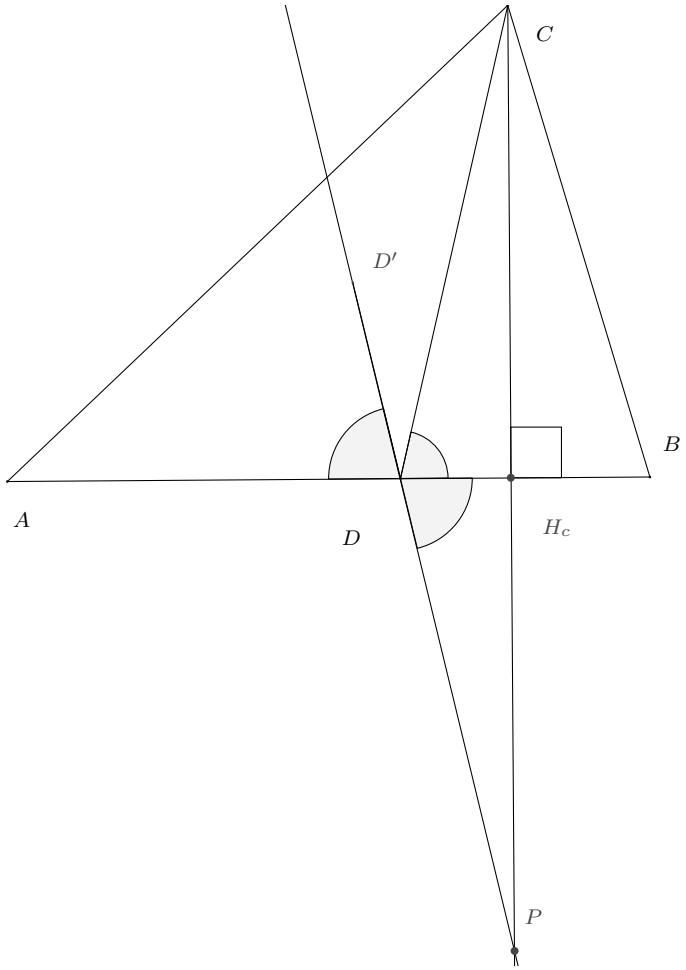
Solution 2.

We have that $\angle ACB = \angle BDE$. Consequently $\angle ADE + \angle ACB = 180^\circ$, i.e. $ADEC$ is inscribed. Then $\angle ECD = \angle EAD$.

Analogously $BCFD$ is inscribed and then $\angle BCD = \angle BFD$, so $\angle BFD = \angle EAD$, meaning that $ADD'F$ is inscribed in a circle. It follows that $\angle FAD' = \angle FDD'$. Also from the fact that $ADEC$ is inscribed in a circle we get $\angle CAE = \angle CDE$ and thus $\angle FDD' = \angle CDE$. But we also have $\angle ADF = \angle BDE$ and then $\angle ADD' = \angle BDC$.



Let CH_c be an altitude in $\triangle ABC$ and P be the intersection point of DD' and CH_c. Then $\angle ADD' = \angle BDP$, i.e. $\angle CDH_c = \angle PDH_c$, but also $DH_c \perp CP$. Thus, H_c is the midpoint of CP. Then P is symmetric to C with respect to H_c and lies on DD' for all D on AB, which completes the proof.



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Solution 3.

Yes. Let $N = 2017$. Any rational $r = \frac{k}{n}$ in reduced form such that $0 < r \leqslant 2$ satisfies

$$N(N+1)|r - N^{\frac{1}{N}}| \geqslant \left(\sum_{m=0}^{N} r^m N^{\frac{N-m}{N}}\right)\left|r - N^{\frac{1}{N}}\right| = \left|r^N - N\right| \geqslant \frac{1}{n^N},$$

the last inequality coming from $k^N \neq Nn^N$.

Now fix n and choose k to be the integer closest to $N^{\frac{1}{N}}n$, which implies $0 < r < 2$, since $0 < N^{\frac{1}{N}} < 2$. Hence,

$$\left|\sin\left((N^{\frac{1}{N}}n - k)\pi\right)\right| \geqslant \frac{2}{N(N+1)n^{N-1}} \geqslant \frac{2}{N(N+1)n^N},$$

as $|\sin x| > \frac{2|x|}{\pi}$ on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (to prove this use that sin is odd and is concave on $[0;\pi]$). Indeed, our choice of k gives $\left|N^{\frac{1}{N}}n - k\right| \leqslant \frac{1}{2}$. Thus, $C = \frac{2}{N(N+1)}$ works for all n.

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Solution 4.

A strategy exists iff N is of the form 2^n. Denote $[N] = \{0, \dots, N-1\}$ and canonically identify a position of the coins with a subset of $[N]$. Let $\mathfrak{P}(X)$ be the set of all subsets of X for any set X. Finally, for a function $h : X \to Y$ the inverse image of a set $Z \subset Y$ is the set $h^{-1}(Z) = \{x \in X | h(x) \in Z\}$.

Assume that such a strategy exists for a given N. Then necessarily for any position of the coins Alice has to flip a different one depending on the label of the magical coin (denoted by k), since otherwise, Bob would reply the same for two different values of k and would be wrong at least in one of the cases. Thus, Alice uses a function $g : [N] \times \mathfrak{P}([N]) \to \mathfrak{P}([N])$ to map the initial conditions (said "init") i.e. the position of all the coins and the label of the magic one to one of the 2^N configurations (said "conf") after her turn. The confs are all reached exactly N times each by the preceding argument.They must then chose a function $f : \mathfrak{P}([N]) \to N$, allowing Bob to map confs to the first coordinate (k) of the init that led to it. In symbols we seek

$$\forall k \, \forall A, \ f(g(k, A)) = k \, .$$

If $N \not| \, 2^N$ i.e. $N \neq 2^n$, this is not possible, because every $k \in [N]$ needs to have 2^N inverse images by the composition $f \circ g$ and, as we saw that all the conf have N inverse images by f, any k has to satisfy $\frac{2^N}{N} \geqslant \left|f^{-1}(\{k\})\right|$ (the number of conf giving k by f). However, summing these inequalities for all k, one finds $2^N \geqslant \left|f^{-1}([N])\right| = |\mathfrak{P}([N])|$, so that all of them must be equalities, and thus $\frac{2^N}{N}$ is a natural number.

Let us now give a winning strategy for $N = 2^n$. To do that identify $[N]$ with $\mathcal{P}([n])$ via binary representation. Set

$$f : \mathfrak{P}(\mathfrak{P}([n])) \to \mathfrak{P}([n]) : A \mapsto \{a \in [n], \, | \, \{B \in A \mid a \in B\} \, | \equiv 1 [\mathrm{mod}\, 2]\}$$

$$g : \mathfrak{P}([n]) \times \mathfrak{P}(\mathfrak{P}([n])) \to \mathfrak{P}(\mathfrak{P}([n])) : (k, A) \mapsto A \Delta \{(f(A) \Delta k)\} \, ,$$

where Δ is the symmetric difference of sets. Then clearly $f(g(k, A)) = k$ and f is indeed the flip of exactly one coin.

Remarque One can find this strategy, easily convincing oneself that changing (two) different bits of the conf has to change (meaning that it becomes different) Bob's response and that in a commutative and associative fashion. What is more, the cardinals of the relevant sets are quite suggestive.

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