Concours Général de Mathématiques «Minko Balkanski»

Solutions

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Solution 1.

$\boxed{2024.}$ Example: $\prod_{i=1}^{2024}(x-i)$.

Assume that there are at least 2025 solutions $(x_i, y_i)_{i=1}^{2025}$ with $x_1 < \cdots < x_{2025}$. Then for any $i \in \{1, \ldots, 1012\}$, we have $1012 < x_{2025} - x_i | y_{2025} - y_i$ and thus $y_i = y_{2025}$. Similarly, $y_1 = y_i$ for $i \geqslant 1014$. Hence, $P(x) = y_1 + a\prod_{i=1}^{1012}(x - x_i)\prod_{i=1014}^{2025}(x - x_i)$ for some integer a. This entails that

$$1012 \geqslant y_{1013} = |P(x_{1013})| \geqslant |a|(1012!)^2 - 1012,$$

so $a = 0$. Contradiction.

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Solution 2.

$\boxed{(5,1).}$ Assume that (x, y) is a solution with $x > 5$ and $y > 1$. Rewrite the equation in the form

$$2^5(2^{x-5} - 1) = 23(23^{y-1} - 1).$$

Since $2^{11} - 1 = 23 \cdot 89$ and 11 is prime, we have that the order of 2 modulo 23 is 11. Thus, 11 divides $x - 5$, so

$$89|2^{11} - 1|2^{x-5} - 1|23(23^{y-1} - 1).$$

In particular, $89|23^{y-1} - 1$. We compute the Legendre symbol

$$\left(\frac{23}{89}\right) = \left(\frac{89}{23}\right) = \left(\frac{20}{23}\right) = \left(\frac{5}{23}\right) = \left(\frac{23}{5}\right) = \left(\frac{3}{5}\right) = -1.$$

Hence, $23^{44} \equiv -1 \pmod{89}$. But Fermat's little theorem gives $89|23^{88} - 1$, so 8 divides the order of 23 modulo 89. It follows that $8|y - 1$ and so $23^8 - 1|23^{y-1} - 1$. It is not hard to check that $2^6|23^8 - 1$, which implies $2^6|23^8 - 1|23^{y-1} - 1|2^5(2^{x-5} - 1)$. This is a contradiction with $x > 5$.

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Solution 3.

$\boxed{\text{No. Yes. Yes.}}$ The density of a box B is the ratio $d(B) = |B \cap A|/|B|$.

(a) Let $c = 1/4$. Take $A = \{(0, 0)\}$.

(b) Let $c = 4/5$. We consider dyadic boxes, starting with a box containing A with density at most $4/5$. If a box B subdivides into $B_1, \ldots, B_4$, we have $\max_i d(B_i) \leqslant 4d(B)$. We take the boxes with density in $[1/5, 4/5)$ and continue to subdivide the others.

(c) Let $c = 2/3$. A box B is bad, if $d(B) \geqslant c$. Consider the subdivision of a box B of size 2^{n+2} with $d(B) < 1/5$ into 16 boxes $(B_{i,j})_{i,j=1}^4$ of size 2^n. If no of these boxes is bad, we add those with density at least $1/5$ to our collection and we continue to subdivide the others. If five or more are bad, we obtain $d(B) \geqslant 5c/16 > 1/5$—contradiction.

Assume that between one and four of the boxes $B_{i,j}$ are bad. If they are all in the same box of size 2^{n+1}, we add a box B' of size $3 \cdot 2^n$ containing it to our collection and we continue to subdivide those among the 7 remaining boxes of size 2^n, whose density is smaller than $1/5$ (we know that they are not bad). Indeed, $d(B') \geqslant c \cdot (4/9) > 1/5$ and $d(B') \leqslant d(B) \cdot (16/9) \leqslant (1/5) \cdot (16/9) < c$.

Finally, assume that there is no box of size 2^{n+1} containing all the bad boxes. Then we can divide B into four boxes of size 2^{n+1}. For each B' among them we have $d(B') \leqslant 4(d(B) - c/16) \leqslant 4(1/5 - 1/24) < c$, since at least one bad box lies outside B'. Thus, we can add B' to our collection if $d(B') \geqslant 1/5$ and continue to subdivide otherwise. It remains to notice that if B is of size 2 and $d(B) < 1/5$, then $B \cap A = \varnothing$ and we do not need to consider this box further.

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